Bound States of Branes with Minimal Energy

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It is pointed out that the energy of the bound states of D-branes and strings is determined by the central charge of the space-time supersymmetry. The universality which is seen at the black hole horizon appears also on the D-brane side: the total energy of the bound states of a given number of branes has a minimum when considered as a function of the independent parameters (moduli). This provides a new evidence that the near-horizon space-time geometry of the dilaton black holes can be represented by the bound states of branes. The axion-dilaton dyonic black holes have the mass formula of the non-threshold type bound state. Upon uplifting to higher dimensions they may give information about such states.

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Supersymmetric General Relativity (SGR) and Quantum Mechanics (QM) have been recently related to each other [1] using the D-brane technology [2], see [3] for a review on black holes in string theory. In SGR one of the main features is the universality property of supersymmetric black holes near the horizon. It reveals in the fact that the 4- and 5-dimensional black holes are attractor systems with fixed points [4] for moduli. The area of the horizon is universally defined by the central charge, extremized in the moduli space [5].

On the other hand, the QM side of the picture so far has not shown any signs of the universality even for the particular black holes for which the geometry of space-time near the horizon is replaced by D-branes and strings.

Here we would like to show how the universality property of supersymmetric black holes near the horizon can be seen on the QM side. Our conjecture will be general, but we will be able to check it only for limited examples available. Consider the supersymmetry algebra

$$\{Q_\alpha, Q_\beta\} = S_{\alpha\beta} \tag{1}$$

in some dimension. Examples in D = 12, 11, 10 and 4 relevant to this discussion can be found in the literature, see [6] for the most recent discussion from the point of view of d=12 and [2] in the context of D-branes. The right-hand side of the algebra contains translation as well as all possible p-form charges permitted by symmetry [7,6,2]. The state with some unbroken supersymmetry is characterized by a vanishing eigenvalue, or eigenvalues of this matrix [8,9,6]

$$\det\,[S_{\alpha\beta}]\mid BPS>=0\ . \tag{2}$$

We will consider here only massive states, which means that there exists a rest frame where the energy of the state which is equal to its mass is given by (in the basis where the eigenvalues of the central charge matrix are real)

$$tr\,[S_{\alpha\beta}]\mid BPS>= H\mid BPS>= E\mid BPS>\ . \tag{3}$$

The SGR part of universality in 4 and 5 dimensions comes from the observation that the central charge in the SUSY algebra is defined by the charge of the graviphoton, which is a particular combination of moduli and quantized charges. Supersymmetric theories with $N > 2$ which have few eigenvalues of the central charge matrix can be understood in terms of N=2 theories. The universality is in the fact that near the horizon the largest eigenvalue of the central charge is extremized in the moduli space, which requires all other eigenvalues to vanish at the attractor. The entropy S of the black hole is given by the volume of the S^2 or S^3 sphere of the radius r_h in d=4 and d=5 respectively [5]:

$$S(d=4) = \frac{A}{4} = \pi (r_h)^2 = \pi (Z\bar{Z})_{\rm fix}\ , \tag{4}$$

$$S(d=5) = \frac{A}{4} = \frac{\pi^2}{2}(r_h)^3 = \frac{\pi^2}{2} Z_{\rm fix}^{3/2}\ . \tag{5}$$

Even more interesting for our purpose here is the fact that simultaneously with defining the entropy, the extremized value of the central charge defines the extremized value of the ADM mass of the black hole [5]:

$$M_{ADM}^{\min}(d=4) = \sqrt{(Z\bar{Z})_{\rm fix}}\ , \tag{6}$$

$$M_{ADM}^{\min}(d=5) = \frac{3\pi}{4} Z_{\rm fix}\ . \tag{7}$$

The central charge is defined by supergravity theory under consideration and depends on moduli ϕ and quantized charges (p, q). In d=4 the central charge, as well as the moduli, is complex, and in d=5 both are real. Upon extremization moduli become fixed functions of charges (p, q) and the entropy as well as the extremized value of the ADM mass of the black hole depend only on charges. It has been proved recently that in N=2 theories with special geometry the extremal value of the supersymmetric black hole mass is the minimal one [10].

$$|Z(\phi,(q,p))| \geq |Z(\phi,(q,p))|_{\frac{\partial |Z|}{\partial \phi}=0} = |Z(\phi_{\rm fix}(q,p),(q,p))|\ . \tag{8}$$

The simplest way to understand how many moduli are fixed at the black hole horizon is to perform a consistent truncation to N=2 theory, starting with N=8,4. Some moduli go into vector multiplets, some into hypermultiplets. Only the first ones are fixed in terms of charges, the scalars of the hypermultiplets are not fixed in terms of charges although in all known supersymmetric black hole solutions they are constants [5]. The number of scalars from the vector multiplets equals to the number of gauge groups minus one since one vector belongs to the gravitational multiplet. In d=4 each gauge group may contain both electric and magnetic charge and each scalar may be complex. In d=5 there are only electric charges in each gauge group and the scalars are real.

Superstring theory supplies the supersymmetry generators Q_α build out from the quantum mechanical states of a string and D-branes. One may try to find the relation between the universal picture in SGR and those examples where the black holes are understood at the QM level.

One may expect that when only superstring excitations, including the D-branes, are taken into account, they may not be sufficient for the complete universal picture at the QM side. One may need to include some QM excitations related to the 2- and 5-branes of the 11-dimensional theory or some other modes. This is indeed the case studied for toroidal compactification in [11] and for more general Calabi-Yau black holes in [12].

Our main conjecture is that in general the QM interpretation of the universal minimization of the central charge is the following. The energy of the bound states of various branes depends on the number of branes and on their energies. The energy-charge of the individual brane states are functions of some independent parameters, which can be identified as moduli, see examples below. The value of the total energy of a given number of brane states is minimal when the energies of the branes pick up the values depending on the number of branes of all kinds, making the bound state with the minimal energy.

Thus we would like to show that the energy of the BPS bound state $E_{\rm bs}$ of various number of branes has the energy which reaches the minimum when the energies of the individual branes $e_{\rm br}$ become particular functions of the numbers of branes $n_{\rm br}$.

$$E_{\rm bs}\left(e_{\rm br}(\phi), n_{\rm br}\right) \geq E_{\rm bs}^{\rm min}\left(e_{\rm br}\left(\phi(n_{\rm br})\right), n_{\rm br}\right) . \qquad (9)$$

The minimization condition is

$$\frac{\partial E_{\rm bs}\left(e_{\rm br}(\phi), n_{\rm br}\right)}{\partial \phi} = 0 . \qquad (10)$$

The minimal value of the energy of the bound state correspond exactly to the ADM mass of the double-extreme black holes with constant moduli [13] when the number of branes is identified with the quantized charges of the black holes. Consider the first example [14], related to 5d black hole: the bound state of some number Q_1 of D-onebranes, Q_5 of D-fivebranes, and Kaluza-Klein momentum in type IIB string theory compactified on T^5. The D-fivebranes are wrapped on T^5. Q_1 D-strings are wrapped along the direction 9 and there is a momentum $P_9 = \frac{N}{R^9}$ along the string which is in direction 9.

Here the 10d Newton constant and 5d Newton constant are related as follows:

$$G_N^{10} = 8\pi^2 g^2 = G_N^5 (2\pi)^5 R_5 R_6 R_7 R_8 R_9 = G_N^5 (2\pi)^5 V R_9 .$$

We are interested in the QM of the near horizon 5d black holes. Therefore in what follows we will fix the 5d Newton constant G_N^5, whereas the 10d Newton constant $G_N^{10} = 8\pi^2 g^2$ will be subject of variation. As long as the radius of 9-th direction R_9 and the string coupling constant are arbitrary, the energy of the BPS bound state described above is (in $\alpha' = 1$ units)

$$E = \frac{R_9 |Q_1|}{g} + \frac{R_9 V |Q_5|}{g} + \frac{|N|}{R_9} = E_1 + E_5 + E_P . \qquad (11)$$

In this example the energy of the bound state is the sum of the energies of the constituents. We are interested in the variation of this energy at the fixed values of the number Q_1 of D-onebranes, Q_5 of D-fivebranes and string states N as well as at the fixed G_N^5. To perform this variation we rewrite eq. (11) as

$$E(g, R_9) = \frac{R_9 |Q_1|}{g} + \frac{g |Q_5|}{4\pi^3 G_N^5} + \frac{|N|}{R_9} . \qquad (12)$$

For simplicity we will fix $4\pi^3 G_N^5 = 1$. We proceed with extremization of the energy as a function of g and R_9, starting with

$$\begin{aligned} E(g, R_9) &= \frac{R_9}{g} |Q_1| + g |Q_5| + \frac{1}{R_9} |N| \\ &= e_1 |Q_1| + e_5 |Q_5| + e_P |N| . \end{aligned} \qquad (13)$$

Variation of the total energy over the string coupling constant gives

$$\frac{\partial E}{\partial g} = -\frac{R_9 |Q_1|}{g^2} + |Q_5| = 0 . \qquad (14)$$

Variation of the total energy over the radius of the 9-th dimension leads to

$$\frac{\partial E}{\partial R_9} = \frac{|Q_1|}{g} - \frac{|N|}{R_9^2} = 0 . \qquad (15)$$

The minimum of the energy of our bound state is reached when the string coupling constant and the radius of the 9-th dimension take the fixed values, prescribed by the integer numbers Q_1, Q_5, N:

$$\begin{aligned} g_{\rm fix} &= \left\{ \frac{|Q_1 Q_5 N|}{|Q_5|^3} \right\}^{1/3} , \\ \left(\frac{1}{R_9}\right)_{\rm fix} &= \left\{ \frac{|Q_1 Q_5 N|}{|N|^3} \right\}^{1/3} , \end{aligned} \qquad (16)$$

and as a consequence,

$$\left(\frac{R_9}{g}\right)_{\rm fix} = \left\{\frac{|Q_1 Q_5 N|}{|Q_1|^3}\right\}^{1/3} . \tag{17}$$

The energy of the bound state at the fixed values of the string coupling constant and the radius of the 9-th dimension in units we have chosen becomes

$$E^{\rm min}(g_{\rm fix}, (R_9)_{\rm fix}) = 3|Q_1 Q_5 N|^{1/3} . \tag{18}$$

It consists of 3 equal contributions: from all D-onebranes, from all D-fivebranes and from all string states:

$$E^{\rm min} = 3\left(\frac{R_9}{g}\right)_{\rm fix} |Q_1| = 3 g_{\rm fix} |Q_5| = 3\frac{|N|}{(R_9)_{\rm fix}} . \tag{19}$$

The black hole entropy is proportional to the minimal energy of the bound state in power 3/2 as expected. Thus in this example we have shown that the minimal energy of the bound state is topological at the fixed values of G_N^5. This is a QM interpretation of the minimization of the black hole area and the black hole ADM mass.

One can rewrite this construction which describes the bound state of D-branes and string states in very special geometry terms adapted to 5d black holes [5] where we identify

$$E = Z = t^I(\phi_1, \phi_2) q_I = t^0 q_0 + t^1 q_1 + t^2 q_2,$$

$$t^0 t^1 t^2 = 1, \tag{20}$$

$$Q_1 = q_0,\ Q_5 = q_1,\ N = q_2,\ \phi_1 = g,\ \phi_2 = R_9.$$

The minimum of the central charge now is identified with the minimum of the bound state energy:

$$Z_{\rm fix} = E^{\rm min}(g_{\rm fix}, R_{\rm fix}) = 3|q_0 q_1 q_2|^{1/3} = 3|Q_1 Q_5 N|^{1/3} . \tag{21}$$

The very special geometry construction is easily generalizable to more general cases when $d_{IJK} t^I t^J t^K = 1$, $I = 0, 1, \ldots, n$. The QM side still has to be found.

Consider the second example [15], related to 4d black hole: the bound state of some number Q_2 of D-twobranes, Q_5 of solitonic fivebranes, Q_6 of D-sixbranes and Kaluza-Klein momentum in type IIB string theory compactified on T^6. Here the procedure is practically the same as in 5d case. One starts with the energy of the bound state consisting of the sum of the energy of the constituents.

$$\begin{aligned} E &= \frac{R_9 R_4}{g}|Q_2| \ + \frac{R_9 V}{g^2}|Q_5| + \frac{R_9 R_4 V}{g}|Q_6| + \frac{1}{R_1}|N| \\ &= E_2 + E_5 + E_6 + E_P \ . \end{aligned} \tag{22}$$

Here $V = R_5 R_6 R_7 R_8$ as before, R_9 is the radius of the 9-th dimension, and R_4 is the radius of the 4-th dimension. We fix the 4d Newton constant $8G_N^4 = \frac{g^2}{R_4 V R_9} = 1$ for simplicity and the bound state energy is:

$$\begin{aligned} E &= \frac{R_9 R_4}{g}|Q_2| \ + \frac{1}{R_4}|Q_5| + g|Q_6| + \frac{1}{R_9}|N| \\ &= E_2 + E_5 + E_6 + E_P \ . \end{aligned} \tag{23}$$

Varying it over R_9, R_4, g we get

$$\frac{R_4}{g}|Q_2| \ - \frac{1}{R_9^2}|N| = 0 \ , \tag{24}$$

$$\frac{R_9}{g}|Q_2| \ - \frac{1}{R_4^2}|Q_5| = 0 \ , \tag{25}$$

$$\frac{R_9 R_4}{g^2}|Q_2| - |Q_6| = 0 \ . \tag{26}$$

The solution for the fixed values of string coupling and dimensions of two circles is

$$\begin{aligned} g_{\rm fix} &= \left\{\frac{|Q_2 Q_5 Q_6 N|}{|Q_6|^4}\right\}^{1/4} , \\ \frac{1}{(R_4)_{\rm fix}} &= \left\{\frac{|Q_2 Q_5 Q_6 N|}{|Q_5|^4}\right\}^{1/4} , \\ \frac{1}{(R_9)_{\rm fix}} &= \left\{\frac{|Q_2 Q_5 Q_6 N|}{|N|^4}\right\}^{1/4} . \end{aligned} \tag{27}$$

Again the useful derivable object is

$$\left(\frac{R_9 R_4}{g}\right)_{\rm fix} = \left\{\frac{|Q_2 Q_5 Q_6 N|}{|Q_2|^4}\right\}^{1/4} . \tag{28}$$

The minimal energy of the bound state is achieved when the contribution from the D-twobranes, solitonic fivebranes, D-sixbranes and string states equal each other:

$$\begin{aligned} E^{\rm min} &= 4\left(\frac{R_9 R_4}{g}\right)_{\rm fix} |Q_2| = 4\left(\frac{1}{R_4}\right)_{\rm fix} |Q_5| \\ &= 4 g_{\rm fix}|Q_6| = 4\left(\frac{1}{R_9}\right)_{\rm fix} |N| \ , \end{aligned} \tag{29}$$

and

$$E^{\rm min} = 4|Q_2 Q_5 Q_6 N|^{1/4} . \tag{30}$$

Relation to the entropy is simple*:

$$S = \pi|Z_{\rm fix}|^2 = \pi(E^{\rm min})^2 = 2\pi|Q_2 Q_5 Q_6 N|^{1/2} . \tag{31}$$

Note that in more general situation when we have 4d black holes of N=2 supergravity with n vector multiplets

*In deriving this relation in the standard form we had to take into account that our units with $8G_N^4 = 1$, which we took for simplicity, has to be changed which provides the correct relation.

the central charge is complex [16,5] and the formula for the energy of the bound state is not given by the sum of the energies of the constituents

$$E = \sum_i E_i \ . \tag{32}$$

In general the energy of the bound state of some QM system (which has to replace the near horizon black hole) is defined by a quadratic relation

$$\begin{aligned} E^2(\phi, \bar{\phi}, p, q) = \sum_{IJ} e^{K(\phi,\bar{\phi})} \Big(& F_I(\phi)\bar{F}_J(\bar{\phi}) p^I p^J \\ & + X^I(\phi)\bar{X}^J(\bar{\phi}) q_I q_J - F_I(\phi)\bar{X}^J(\bar{\phi}) p^I q_J \\ & - X^I(\phi)\bar{F}_J(\bar{\phi})\ q_I p^J \Big), \end{aligned} \tag{33}$$

and the minimal energy is given when the moduli take the fixed values $\phi_{\rm fix}(p,q)$ defined by extremization of the energy.

$$E^{\min}(p,q) = E(\phi, \bar{\phi}, p, q)_{\rm fix} \ . \tag{34}$$

Only in special cases when the central charge is real the energy is given by a sum of the energies of constituents whose numbers are p^I or q_I, and the number of moduli is n:

$$E = \sum_{I=0}^{n} e^{\frac{K(\phi)}{2}} \left(X^I(\phi) q_I - F_I(\phi) p^I \right) \ . \tag{35}$$

The condition when this happens is that in a given gauge group either p^I or q_I are vanishing. If at least in one gauge group both q and p do not vanish, the energy is not the sum of the energies of constituents. All examples of black holes-D-branes above obviously fall into the class of bound states with sum of energies, or so called threshold bound states. Indeed they are dilaton-no-axion solutions. The same is true for the CY no-axion black holes of [17] which have been recently understood by counting open 2-branes which end on the M-5-brane in [12].

The non-threshold bound states eventually will be identified with the dyonic black holes which have electric and magnetic charges in some gauge groups and complex moduli, i.e. both dilatons as well as axions are present in the solutions. Many such black hole solutions are known in SGR, see [18,17]. The simplest one is the axion-dilaton $SL(2,Z)$-invariant solution [18]. The duality invariant mass formula for these dyonic black holes is always of the type shown in eq. (33). It remains to find their QM description. One may expect that the uplifting of such black holes to d=10, 11 will explain the structure of such bound states, where the energy of the total state is not a sum of energies of the constituents. It is, however, guaranteed that for the regular black holes with the singularity covered by the horizon the energy will have a minimum, related to the axion-dilaton black hole entropy.

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